Lecture 7. Autonomous Equations

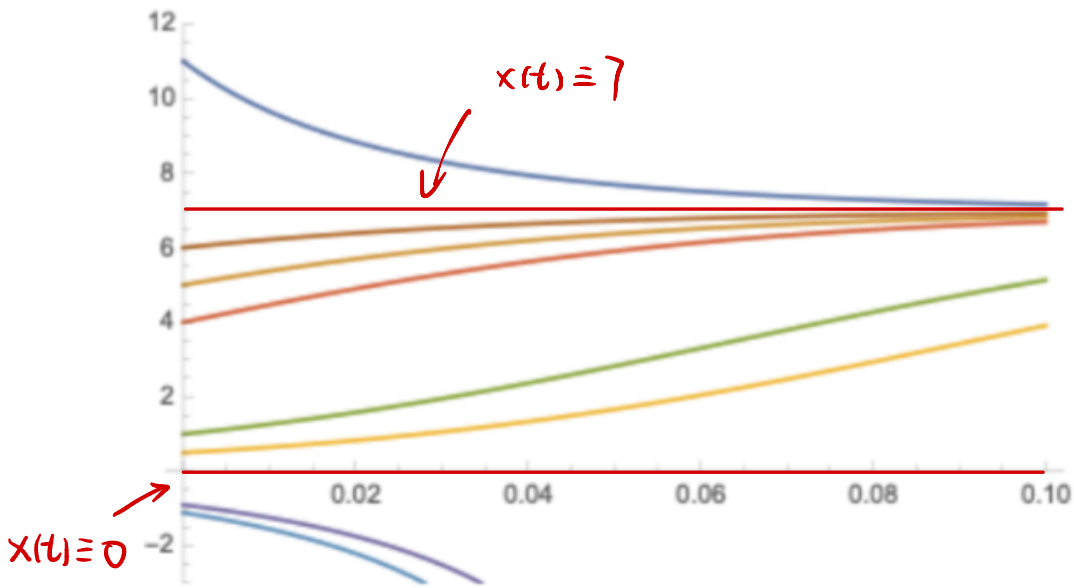
Example Consider the following differential equation

$$\frac{dx}{dt} = 4x(7-x)$$

Solving this differential equation, we get the following solutions:

- General solutions $x(t) = \dfrac{7}{1+Ce^{-28t}}$
- Singular solutions $x(t) \equiv 0$ and $x(t) \equiv 7$.

We sketch the some solution curves for the equation:



In this section, we will discuss the solutions $x(t) \equiv 0$ and $x(t) \equiv 7$ and their stability.

Critical points and equilibrium solutions

Let's consider the differential equation of the form

$$\frac{dx}{dt} = f(x) \tag{1}$$

- We call it an **autonomous** first order equation--one in which the independent variable t does not appear explicitly.
- The solutions of the equation $f(x) = 0$ play an important role and are called **critical points** of the autonomous differential equation $\dfrac{dx}{dt} = f(x)$.
- If $x = c$ is a critical point of Eq. (1), then the differential equation has the constant solution $x(t) \equiv c$.
- A constant solution of a differentail equation is sometimes called an **equilibrium solution**.

Stability of Critial Points

Consider the differential equation of the form

$$\frac{dx}{dt} = f(x) \tag{3}$$

- **Stable:** In general, a critical point $x = c$ of an autonomous first-order equation is said to be **stable** if the initial value x_0 is sufficiently close to c, then $x(t)$ remains close to c for all $t > 0$.
- **Unstable:** The critical point $x = c$ is **unstable** if it is not stable.
- **Semi-stable:** The critical point $x = c$ looks stable on one side but unstabe on the other side. See **Example 1** below. unstable

Funnels and Spouts

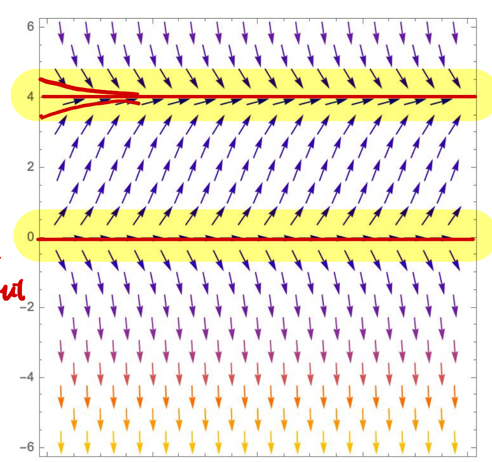
For example, consider $\dfrac{dx}{dt} = 4x - x^2$. The vector field is as below. $f(x) = 4x - x^2 = x(4-x)$

thus $x=0$ and $x=4$ are critical points.

Using the vector field, we can graph the solution curves:

x=4 stable acts like a funnel

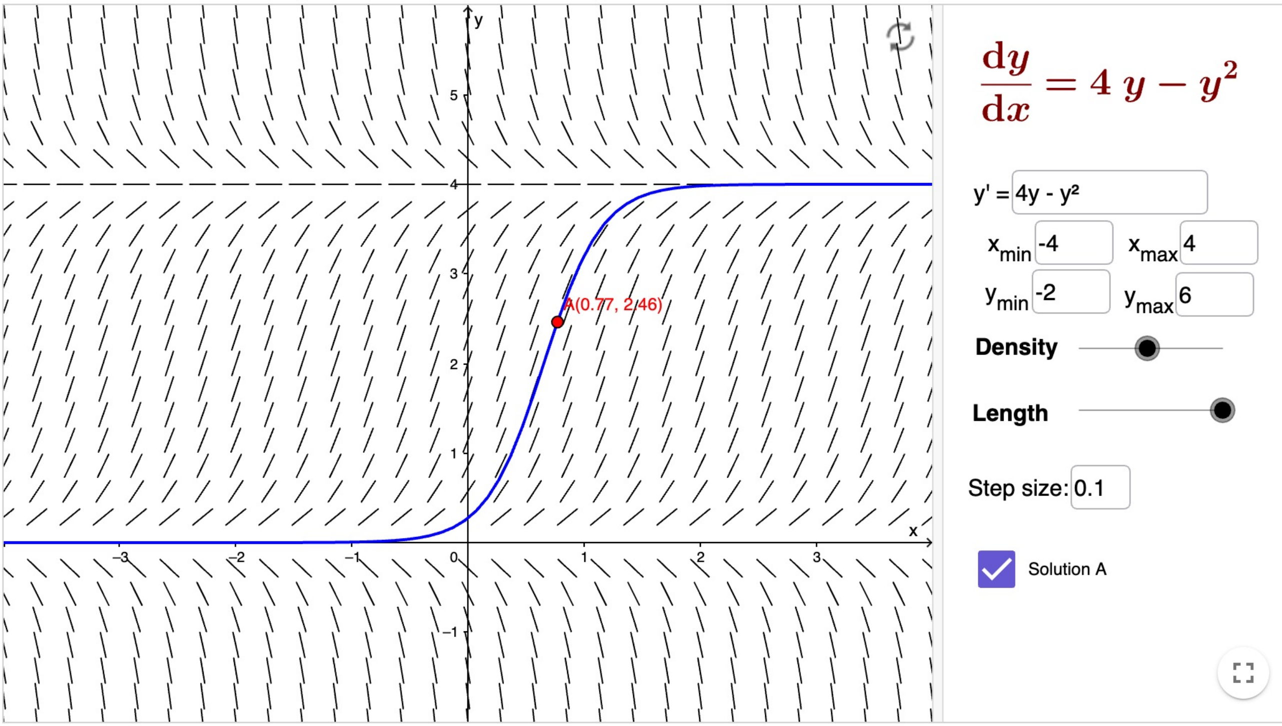
x=0 unstable acts like a spout



Slope field generator:

https://www.geogebra.org/m/Pd4Hn4BR

This applet will generate Direction Fields and approximate solution curves given initial values.
Click and drag the initial point A to see its corresponding solution curve Credits: Originally created by Chip Rollinson



Phase diagram

We can summarize the behavior of the solution by the phase diagram, which indicates the direction (phase) of change of x as a function of x itself.

Example 1. Calculate the critical points of the following equation and draw the phase diagram. Classify the stability of the critical points.

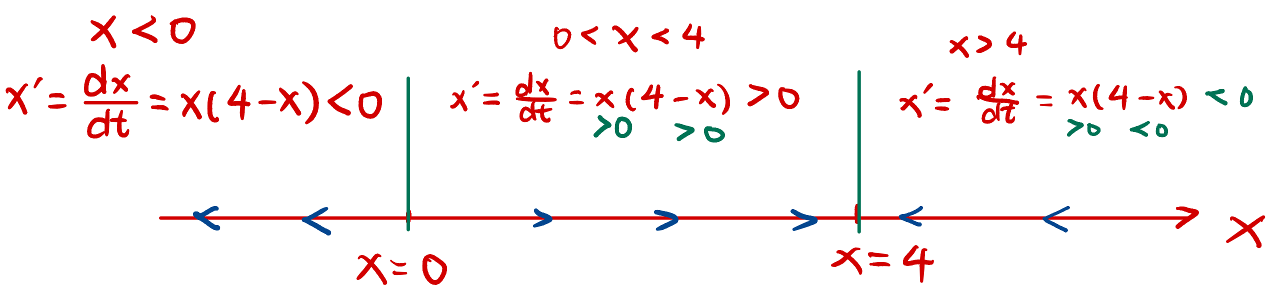
$$x'(t) = \frac{dx}{dt} = 4x - x^2$$

ANS: Let $f(x) = 4x - x^2 = x(4-x) = 0$

$\Rightarrow$ $x=0$ and $x=4$ are critical points.

We construct the phase diagram as the following:

First we draw the x-axis with the critical points



Stable critical point(s): $x = 4$

Unstable critical point: $x = 0$

Example 2.

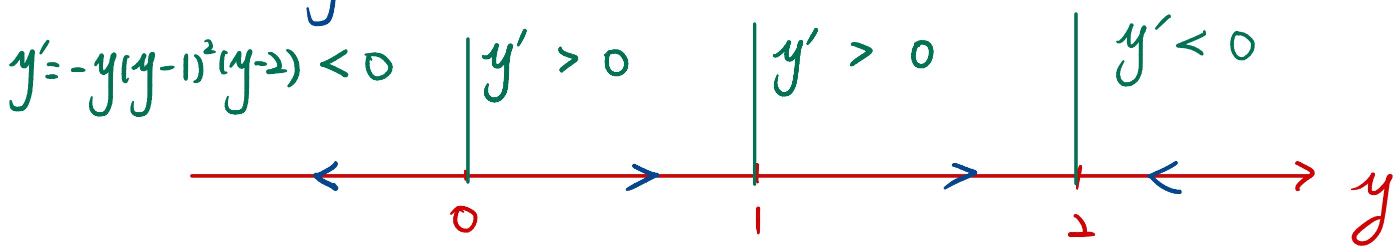
Consider the initial value problem, $y' = -y(y-1)^2(y-2), \quad y(0) = y_0.$

(1) Calculate the critical points of this equation and draw the phase diagram. Classify the stability of the critical points.

ANS: Let $f(y) = -y(y-1)^2(y-2) = 0 \Rightarrow y=0,\ y=1$ or $y=2$.

Thus $y = 0, 1, 2$ are critical points.

Phase diagram



Thus

stable critical point(s): 2

unstable critical points: 0, 1

semistable

(2) For which initial value y_0 below is $\lim_{t\to\infty} y(t) = 1$?

Your answer:___B

A. -0.001

$\boxed{\textbf{B. } 0.001}$

C. 1.001

D. 2.001

E. None of the above

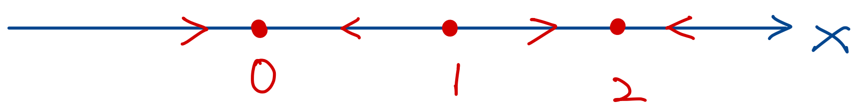
Exericse 3. Calculate the critical points of the equation below and draw the phase diagram. Classify the stability of the critical points.

$$x' = -4x(1-x)(2-x) = f(x)$$

ANS: Let $f(x) = -4x(1-x)(2-x) = 0$, we have $x=0$, $x=1$, $x=2$.

Thus the critical points are $0, 1, 2$.

Phase diagram:



- When $x<0$, $x' = -4x(1-x)(2-x) > 0$
- When $0<x<1$, $x' = -4x(1-x)(2-x) < 0$
- When $1<x<2$, $x' = -4x(1-x)(2-x) > 0$
- When $x>2$, $x' = -4x(1-x)(2-x) < 0$

From the phase diagram, we know we have
stable critical point $0, 2$.
unstable critical point 1.

Exericise 4. Find an autonomous differential equation with all of the following properties:

- equilibrium solutions at $y=0$ and $y=7$, ①
- $y' > 0$ for $0 < y < 7$, and ②
- $y' < 0$ for $-\infty < y < 0$ and $7 < y < \infty$. ③

ANS: ① implies $y = 0, 7$ are critical points to $f(x)$.

By ② and ③, we know

$$y' = y(7-y)$$

is an autonomus diff. eqn satisfies ①–③.

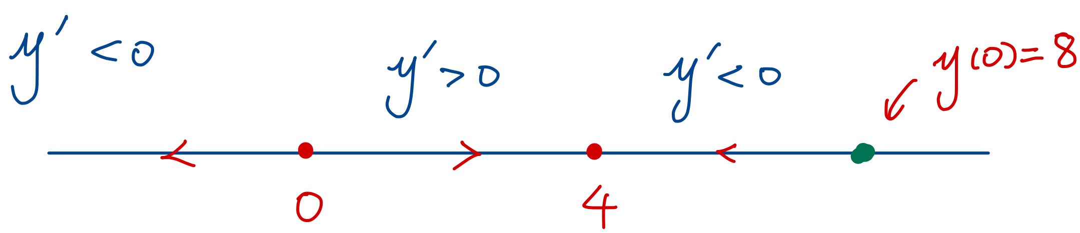
Exercise 5. Let $y(t)$ be a solution of $y' = \frac{1}{4}y\left(1 - \frac{y}{4}\right)$ such that $y(0) = 8$.

Determine $\lim_{t\to\infty} y(t)$ without finding $y(t)$ explicitly.

ANS: This can be answered by checking the phase diagram.

The critical points are $y = 0, 4$.

The phase diagram:



Thus if $y_0 > 4$ or $0 < y_0 < 4$, $\lim_{t\to\infty} y(t) = 4$